

## Non-Commutative Calabi-Yau Manifolds

David Berenstein,<sup>\*</sup>

and

Robert G. Leigh<sup>†</sup>*Department of Physics, University of Illinois, Urbana, IL 61801*

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**Abstract**

We discuss aspects of the algebraic geometry of compact non-commutative Calabi-Yau manifolds. In this setting, it is appropriate to consider local holomorphic algebras which can be glued together into a compact Calabi-Yau algebra. We consider two examples: a toroidal orbifold  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , and an orbifold of the quintic in  $\mathbb{CP}_4$ , each with discrete torsion. The non-commutative geometry tools are enough to describe various properties of the orbifolds. First, one describes correctly the fractionation of branes at singularities. Secondly, for the first example we show that one can recover explicitly a large slice of the moduli space of complex structures which deform the orbifold. For this example we also show that we get the correct counting of complex structure deformations at the orbifold point by using traces of non-commutative differential forms (cyclic homology).

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<sup>\*</sup>e-mail: [berenste@pobox.hep.uiuc.edu](mailto:berenste@pobox.hep.uiuc.edu)

<sup>†</sup>e-mail: [rgleigh@uiuc.edu](mailto:rgleigh@uiuc.edu)

# 1 Introduction

Studying D-branes by the boundary state formalism is in general difficult. In particular one needs an exact closed string conformal field theory (for reviews, see [1, 2]). Although these are powerful techniques, once one deforms the theory, the boundary states may be difficult to follow through the deformation. One can also study D-branes on smooth manifolds in the large volume limit by means of algebraic geometry, but there are known examples of string compactifications where singularities cannot be resolved completely (for example, orbifolds with discrete torsion[3]). In this case a commutative algebraic geometric resolution gives the wrong counting of deformations. This signals that commutative algebraic geometry methods can be deficient in extracting the proper geometry for a string background which is singular.

In previous work [4, 5] we have shown that following marginal and relevant supersymmetric deformations of a conformal field theory associated to the near horizon geometry of a collection of  $D3$  branes gives rise to non-commutativity in the moduli space of vacua. These field theories are related to orbifolds with discrete torsion, and we began work on a systematic treatment of non-commutativity from a holomorphic algebro-geometric point of view. The objective of the present letter is to show several examples of *compact* non-commutative geometries. We will show that this provides the proper set of tools for describing the aforementioned singularities. The non-commutativity is inherent and may not be removed by deformations. The process of compactification is a prescription for glueing together local non-commutative algebras.

In Section 2 we review the formulation of non-commutative algebraic geometry used in [4]. In Sections 3,4 we give two examples; these are orbifolds with discrete torsion of (a) a 6-torus and (b) the quintic in  $\mathbb{CP}_4$ . The first example is treated in more detail as we can calculate exactly the global deformations of the algebra, although due to some simplifying choices only a certain class of these are studied directly here. These are seen to be associated to deformations of the complex structure of the Calabi-Yau space (namely, the holomorphic constraints on the non-commutative algebra), and we find agreement with the results of Vafa and Witten [3] in the sense that all of the deformations are allowed by closed string considerations. At the orbifold point, all local deformations are accounted for by traces of differential forms (*i.e.*, by cyclic homology).

## 2 An approach to non-commutative algebraic geometry

In this section we will give a condensed version of the geometrical ideas described in [4]. The idea is to begin with a (non-commutative) algebra  $\mathcal{A}$  over  $\mathbb{C}$  that is taken to be the (local) algebra of holomorphic functions on a (non-commutative) algebraic space, which we label  $\mathcal{M}_{\mathcal{A}}$ . The center of the algebra  $\mathcal{A}$ , labelled  $\mathcal{Z}\mathcal{A}$ , is the subalgebra consisting of elements of  $\mathcal{A}$  which commute with  $\mathcal{A}$ .

The center is a commutative algebra, and as such describes an affine space  $\mathcal{M}_{\mathcal{Z}\mathcal{A}}$ . The natural inclusion of the center into the algebra is to be taken as pullback of functions, so we

have a map

$$\mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{M}_{\mathcal{Z}\mathcal{A}} \quad (1)$$

This local algebra as given is not a  $C^*$  algebra, as there is no complex conjugation, only holomorphic variables. We can make a  $C^*$  algebra if we introduce the complex conjugate variables to  $\mathcal{A}$ , and make non-holomorphic combinations which are bounded. Thus in principle we can construct an associated system of local  $C^*$  algebras which can be embedded into a larger  $C^*$  algebra  $\tilde{\mathcal{A}}^*$ . An example is the usual quantum torus: there is the algebraic relation  $UV = qVU$ , and additional unitarity conditions  $U^{-1} = U^\dagger$  and  $V^{-1} = V^\dagger$ .

A central idea is that there are two geometries: a commutative geometry on which closed strings propagate, and a non-commutative version for open strings [6]. Here, we identify the commutative space with the center  $\mathcal{Z}\mathcal{A}$  of the algebra. The center of the algebra describes locally a Calabi-Yau manifold with singularities which is also taken to be large so that a semiclassical string analysis is valid. As there are singularities, the commutative geometry alone does not describe how these are resolved, and it is here that the non-commutative geometry gives the extra information needed to provide the boundary conditions for twisted states. Thus, the twisted states will be sensitive to the non-commutative geometry at the singularities. We will give a description of the massless closed string states that count complex structure deformations directly with non-commutative geometry tools, as such we will not compute vertex operators for twisted states directly. We will show that our definition gives the right counting of states, but the connection between the two possible descriptions is not clear.

Point-like  $D$ -branes on the geometry associated to  $\mathcal{A}$  are constructed algebraically as irreducible representations of  $\mathcal{A}$ , where two representations related by conjugation by  $GL(n, \mathbb{C})$  are identified. We will require that these irreducible representations are finite dimensional and thus given by matrices, and moreover that the dimension associated to the center of the algebra is the compactification dimension.

On each of these representations Schur's lemma forces the elements of the center to be proportional to the identity, and as such the representation defines a point in the commutative geometry of the center. Thus we can say where on the Calabi-Yau space the point like  $D$ -brane is located.

It is important that the point-like  $D$ -branes can probe all of the geometry of the Calabi-Yau space. Algebras which satisfy these conditions are referred to as *semi-classical*. The quantum torus, at rational values of  $B$ , is an example of a semi-classical  $C^*$  algebra. In this case,  $U^{\pm n}$  and  $V^{\pm n}$  generate the center of the algebra, where  $q^n = 1$ .

In the examples studied in [4], these requirements seem to imply that the algebra  $\mathcal{A}$  is finitely generated as an algebra over  $\mathcal{Z}\mathcal{A}$ . As such, there is an upper bound on the dimension of the irreducible representations of  $\mathcal{A}$ , and the representations that satisfy this upper bound are taken to describe a point-like brane in the bulk. On taking limits bulk representations can become reducible and this phenomenon is interpreted as brane fractionation at singularities.

More generally, we can expect to build holomorphic  $D$ -branes as coherent sheaves over the ring  $\mathcal{A}$ . As the center acts on  $\mathcal{A}$ , these also have an interpretation as coherent sheaves over the commutative structure, and one can ask if given a coherent sheaf over the center

whether it has an action of  $\mathcal{A}$  which makes it a coherent sheaf on the non-commutative space. It may be the case that an obstruction to such a lifting occurs. This would be interpreted as a breaking of the brane in the non-commutative geometry, and as such it does not exist. This is also equivalent to the anomaly in brane charge calculated by Witten for orbifolds with discrete torsion [7, 8].

Finally, given these algebras one would like to construct closed string states. These are conjectured to be associated with single-trace operators including differential forms. This is motivated by ideas in the AdS/CFT correspondence [9, 10, 11]. The trace is necessary to render the operator gauge invariant, and differential forms arise in thinking in terms of vertex operators for the closed strings. This is the construction we are looking for, where there is a description of closed string states directly from the non-commutative algebra point of view, which might be generalized to situations where there are no other constructions available for the closed string states. We define the support of a closed string state as the set of points of the non-commutative geometry where the trace of the state does not vanish. We will see later in this paper, that this conjecture identifies closed strings with cyclic homology and that moreover we get the right counting of states.

Note that most string states on a space are not holomorphic, thus one needs to introduce a  $C^*$  algebra by adding the complex conjugates of the variables and finding the complete set of commutation relations. One must do this in such a way that the holomorphic center is in the center of the full algebra, and that the representation theory of the holomorphic local algebra gives the representation theory of the full  $C^*$  algebra at the same point (perhaps with some subtleties at singularities). In a field theory context, this second part amounts to solving the  $D$ -terms of the theory.

### 3 Orbifold of the torus

Consider the orbifold  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  with discrete torsion. The non-compact version has been studied extensively.[12, 13, 14] Here, we will use the compact orbifold as a target space for D-branes and show that the orbifold can be regarded as a non-commutative Calabi-Yau compact space.

The first thing we need is a model for the orbifold, so we choose to represent  $T^6$  as the product of three elliptic curves, each given in Weierstrass form

$$y_i^2 = x_i(x_i - 1)(x_i - a_i) \quad (2)$$

with a point added at infinity for  $i = 1, 2, 3$ . The  $\mathbb{Z}_2 \times \mathbb{Z}_2$  will act by  $y_i \rightarrow \pm y_i$  and  $x_i \rightarrow x_i$  so that  $y_1 y_2 y_3$  is fixed under the orbifold action. This is necessary in order to satisfy the CY condition on the quotient space. The four fixed points of the orbifold at each torus are located at  $y_i = 0$  and at the point at infinity. This point at infinity can be brought to a finite point by a change of variables

$$\begin{aligned} y_i \rightarrow y'_i &= \frac{y_i}{x_i^2} \\ x_i \rightarrow x'_i &= \frac{1}{x_i} \end{aligned} \quad (3)$$

With these two patches, we can cover each elliptic curve of the product.

Now we would like to introduce discrete torsion. It is necessary that around each of the singularities the local algebra of holomorphic functions agree with the non-compact case studied in [12, 13, 14]. This may be accomplished by choosing the  $y_i$  to be anti-commuting variables

$$y_i y_j = -y_j y_i \quad \text{for } i \neq j \quad (4)$$

and the  $x_i$  to be in the center of the algebra, appropriate to  $\mathbb{Z}_2$  discrete torsion.

From this starting point, we find that  $w = y_1 y_2 y_3$  is in the center of the algebra. Thus the invariant variables of the orbifold are exactly the same variables which belong to the center of the local algebra, and the center of the algebra reproduces the orbifold space as a commutative algebra. Now consider transforming to another patch by (3). Note that since  $x_i$  is in the center of the algebra, there is no ordering ambiguity, and the structure of (4) does not change.

To find the points of the compact non-commutative geometry, we consider the representation theory of the non-commutative holomorphic algebra locally. At a generic (bulk) point, there is one irreducible representation of the non-commutative algebra for each commutative point in the quotient space. Indeed, the following represents the algebra with discrete torsion

$$y_i = b_i \sigma_i \quad (5)$$

where the  $b_i$  are complex scalars and the  $\sigma_i$  are the Pauli matrices. There is a multiplicity of inequivalent representations here, one for each set of roots  $x_i$  of the Weierstrass forms.

At the fixed planes, this representation becomes reducible as two out of the three  $y_i$  act by zero. Thus we get two distinct non-commutative points, as there are two different irreducible representations corresponding to the two eigenvalues of the non-zero  $y_k$ .

By taking this limit of the representation, we see that the branes fractionate on reaching the singularity. The non-commutative points of the singular planes are then seen to be a double cover of the commutative singular plane, which is a  $\mathbb{CP}^1$ . The double cover is branched around the four points  $x_k = 0, 1, a_k, \infty$  and hence the non-commutative points form an elliptic curve of the form (2). Around each of these four points there is a  $\mathbb{Z}_2$  monodromy of the representations, which is characteristic of the local singularity as measuring the effect of discrete torsion [14].

### 3.1 Global Deformations of the orbifold

If we are to identify this non-commutative geometry with the orbifold with discrete torsion, we should in principle be able to reproduce the deformations of the orbifold. These were calculated in [3] using closed string methods. With this in mind, we must understand how to modify the algebraic relations so that we get a new geometry which describes this deformation. These correspond to global deformations; in subsection 3.2, we will then consider infinitesimal deformations of the orbifold.

We proceed by studying the possible deformations of (4) within a local coordinate patch. The deformation must be of the form

$$y_i y_j + y_j y_i = c(\mathcal{Z}\mathcal{A}) \quad (6)$$

where on the right-hand side we have a function of the center of the local algebra. This deformation breaks quantum symmetries, so as shown in Ref. [14], they are associated with vevs of twisted string states.

As remarked in [3], the generic deformation of the Calabi-Yau manifold will be a double cover of  $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ . Thus, we should have three variables  $x_1, x_2, x_3$  in the center of the deformed algebra. In the following, to simplify matters, we will assume that the  $y_i$  are given by their Weierstrass form. This restricts the deformations to a certain class which is not generic; there are other deformations which are not in this class that we will not discuss here (but see Section 3.2).

As we are doing algebraic geometry, it must be the case that the right hand side of (6) is polynomial and free of poles in each patch. Specifically, (6) may only take the form

$$y_1 y_2 + y_2 y_1 = 2P_{12}(x_1, x_2, x_3) \quad (7)$$

where  $P_{12}$  is a polynomial. Under the change of variables (3) on  $x_1, y_1$ ,  $P_{12}$  transforms as

$$P_{12}(x_1, x_2, x_3) \rightarrow x_1^2 P_{12}(1/x_1, x_2, x_3) \quad (8)$$

Thus, as  $P_{12}$  should transform into a polynomial,  $P_{12}$  must be of degree at most two in  $x_1$ . Similarly, it is at most degree two in  $x_2$  and independent of  $x_3$ . A similar result holds for the other two commutation relations, so we get two more polynomials  $P_{13}(x_1, x_3), P_{23}(x_2, x_3)$ . Each of these polynomials has nine parameters.

Now, one sees that  $y_1 y_2 y_3$  does not belong to the center of the local algebra anymore, and needs to be modified. To see what this modification is, we compute

$$[y_1 y_2 y_3, y_1] = y_1 y_2 (y_3 y_1 + y_1 y_3) - y_1 y_2 y_1 y_3 - y_1 y_1 y_2 y_3 \quad (9)$$

$$= 2y_1 y_2 P_{13} - 2y_1 y_3 P_{12} \quad (10)$$

$$= [y_1, y_2] P_{13} - [y_1, y_3] P_{12} \quad (11)$$

Similar calculations with  $y_2, y_3$  are sufficient to see that

$$w = y_1 y_2 y_3 + P_{13} y_2 - P_{23} y_1 - P_{12} y_3 \quad (12)$$

belongs to the center of the local algebra.

Now it is simple to see that  $w^2$  is given by a polynomial of degree 4 in  $x_1, x_2, x_3$ . When we do a change of variables to another coordinate patch as in (3), then  $w$  transforms simply  $w \rightarrow \frac{w}{x_i^2}$  and thus correctly describes a double cover of  $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ .

The total number of deformations that we account for in this way is 30, nine from each polynomial  $P_{ij}$ , plus the three  $a_i$  that describe the  $y_i$  as a double cover of the  $x_i$ . The total number of deformations is actually 51 [3] and so we are missing a number of complex structure deformations.

Indeed, for each pair  $P_{ij}$  one should be able to count  $16 = 4 \times 4$  deformations instead of  $9 = 3 \times 3$ . Notice that we are missing one deformation for each of the  $y_i$ ; this suggests that in general we should not be able to interpret the  $y_i$  locally as a double cover of each of the  $\mathbb{CP}^1$ 's, so our choice of a simple form for the  $y_i$  in terms of the  $x_i$  is the reason for the mismatch.

Nevertheless, the deformations that we have considered do resolve all of the codimension two singularities. Locally the analysis is the same as in Ref. [12], so there is a conifold singularity for each of the codimension three singularities of the orbifold.

### 3.2 Local Deformations and Cyclic Homology

Now, let us analyze the possible infinitesimal variations of complex structure at the orbifold point. For commutative Calabi-Yau manifolds, these are given by holomorphic sections of  $\Omega^{2,1}(\mathcal{M})$ , the  $(2,1)$  forms. That is, we need exact forms of the type

$$f d\phi^i \wedge d\phi^j \wedge d\bar{\phi}^k \quad (13)$$

This notion must be modified appropriately for non-commutative geometry. The deformations of complex structure are related to closed string states, and from the AdS/CFT correspondence, we expect that these in turn correspond to single-trace operators. Thus the natural modification is to include a trace; that is, we consider single trace operators on the module of differentials. This is precisely the type of objects that appear in cyclic cohomology [15].

We will study this cohomology directly within the representation (5). Thus, we write

$$dy_i = \sigma^i db_i \quad (14)$$

and we take complex conjugation on the  $b_i$  as the  $*$ -conjugation of the  $y_i$ . That is,  $y_i$  commutes with its adjoint  $\bar{y}_i$ , and anticommutes with the adjoints of the other  $y_k$ .

Now, we are looking for elements of  $\Omega^{2,1}(\mathcal{M})$  which are associated with codimension two singularities. A standard way to count in ordinary orbifolds is to blow up the singularities, and then consider the cohomology of the smooth manifold. Since the singularities are resolved by 2-spheres, we may think of a given  $(2,1)$ -form as giving, upon integration over the  $S^2$ , a holomorphic one-form.

Thus in our algebraic setting, to study  $\Omega^{2,1}$ , we should look for one-forms such as

$$\text{tr} \frac{dy_1}{x_1} = \frac{1}{x_1} d \text{tr} y_1 \quad (15)$$

Notice that these are proportional to  $\sigma_1$  and hence the trace vanishes at generic points. As we go to a singularity, say  $y_2 = y_3 = 0$ , we should take the trace in distinct irreducible representations of the algebra. As the branes fractionate at this singularity,  $\sigma_1$  decomposes into two one-dimensional irreducible representations and (15) is non-zero. The obvious interpretation is that the  $(2,1)$ -form has support only at the singularity – the above closed string state couples to fractional branes, and thus should be interpreted as a twisted sector of the string.

We may repeat this analysis at each of the singular planes; at each, we get one holomorphic form of the right type which is supported over the singularity. This construction should be interpreted locally, and not as giving a single global form on all the planes simultaneously. In this fashion we get a candidate for each complex deformation in the (non-commutative) cohomology of differential forms of the space.

Moreover, notice that the support of the closed string state depends on the branes fractionating at the singularities. Since the branes don't fractionate further at the codimension three singularities, we don't expect to see any new (massless) closed string states that are supported at these singularities. There seems to be no complex structure moduli supported in codimension three. We conclude that the non-commutative geometry is capturing all of the string theory data of the orbifold.

## 4 Orbifold of the Quintic

Now, we consider a second example, a non-commutative version of (an orbifold of) the quintic in  $\mathbb{CP}^4$  [16, 17, 18]. This example allows us to show how to glue together patches in a situation less trivial than the toroidal orbifold.

The complex structure moduli space of the quintic has special points with a  $\mathbb{Z}_5^3$  group of isometries. In this case, we may consider the orbifold constructed by modding out these symmetries. This orbifold may also include discrete torsion phases. As we will see, we will actually succeed in taking only a  $\mathbb{Z}_5 \times \mathbb{Z}_5$  orbifold, but this is sufficient to demonstrate the principles involved.

We begin with the quintic described by

$$\mathcal{P}(z_j) = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + \lambda z_1 z_2 z_3 z_4 z_5 = 0 \quad (16)$$

where the  $z_j$  are homogeneous coordinates on  $\mathbb{CP}^4$ . The  $\mathbb{Z}_5^3$  action is generated by phases acting on the  $z_j$  as  $z_j \rightarrow \omega^{a_j} z_j$ , with  $\omega^5 = 1$ , and the vectors

$$\vec{a} = (1, -1, 0, 0, 0) \quad (17)$$

$$= (1, 0, -1, 0, 0) \quad (18)$$

$$= (1, 0, 0, -1, 0) \quad (19)$$

consistent with the CY condition  $\sum a_i = 0 \bmod 5$ . We have chosen the action such that  $z_5$  is invariant. This allows us to consider a coordinate patch where  $z_5 = 1$ , and this patch is invariant under the group action. Within this coordinate patch, we will be interested in the local non-commutative algebra of the other four variables. Later, we will consider transformations to other patches.

The discrete torsion is classified by [19]

$$H^2(\mathbb{Z}_5^3, U(1)) = \mathbb{Z}_5^3 \quad (20)$$

so we need three phases to determine the geometry.

The invariant quantities (within the coordinate patch) are given by  $z_i^5$  and  $z_1 z_2 z_3 z_4$ . We will require that these remain in the center of the non-commutative version of the quotient space. Experience with standard orbifolds with discrete torsion suggests that we take the



variables to commute up to phases as follows

$$z_1 z_2 = \alpha z_2 z_1 \quad (21)$$

$$z_1 z_3 = \beta \alpha^{-1} z_3 z_1 \quad (22)$$

$$z_1 z_4 = \beta^{-1} z_4 z_1 \quad (23)$$

$$z_2 z_3 = \alpha \gamma z_3 z_2 \quad (24)$$

$$z_2 z_4 = \gamma^{-1} z_4 z_2 \quad (25)$$

$$z_3 z_4 = \beta \gamma z_4 z_3 \quad (26)$$

with  $\alpha, \beta, \gamma$  being fifth roots of unity. Notice that we have three fifth roots of unity to choose from, which gives us the same number of choices as expected from (20). We will consider the generic case where all three  $\alpha, \beta, \gamma$  are different from one and we want to find the irreducible representations of this algebra within the patch  $z_5 = 1$ .

It is simple to see that there are five-dimensional representations. Indeed, consider the matrices

$$P = \text{diag}(1, \alpha, \alpha^2, \alpha^3, \alpha^4), \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (27)$$

$\alpha$  is a fifth root of unity, and so there are actually five distinct 5-dimensional representations here. In terms of the matrices  $P, Q$ , we can identify  $z_1 = b_1 P$  and  $z_2 = b_2 Q$ , where  $b_j$  are arbitrary complex numbers. Similarly, we may write  $z_3 = b_3 P^m Q^n$  and  $z_4 = b_4 P^{-m-1} Q^{-n-1}$ , where  $m, n$  are determined by the phases  $\beta, \gamma$  (i.e.,  $\beta = \alpha^{n+1}$  and  $\gamma = \alpha^{m+1}$ ). These choices guarantee that  $z_1 z_2 z_3 z_4 z_5$  is in the center of the algebra.

The defining relation (16) requires that

$$\sum_{i=1}^4 b_i^5 + \lambda \prod_{i=1}^4 b_i = -1 \quad (28)$$

As given, the representation becomes reducible only when three out of the four  $z_i$  act by zero, which signals that there are no fractional branes for the (would be) generic codimension two singularities. Indeed one can see that this is the case locally, as there are other invariant variables in  $z_1, z_2, z_3$  apart from  $z_1^5, z_2^5, z_3^5$ , which signals that we have only done a  $\mathbb{Z}_5$  orbifold locally.

At co-dimension three singularities, we do get the expected number of fractional branes however, which is five, so our analysis agrees with a direct analysis local to the singularity [20]. We also recover the local quiver algebra by noticing that at least one of the  $z_j$  has a large vev (see eq. (28)), and this field is responsible for breaking the gauge group from  $U(5)$  to  $U(1)^5$ . Also, ensuring that the commutation relations with this large field hold forces many of the matrix terms in the other fields to vanish, thus giving the correct local data to reconstruct the quiver diagram of the orbifold.

This fact can be traced back to how the discrete torsion lattice is related to the orbifold group. Indeed, we expect D-brane states to be given by a formal quiver diagram corresponding to the irreducible projective representations of the group  $\Gamma$  with a given cocycle [12, 21]. No matter what choice we make for the cocycle condition, as long as it is not trivial, there are 5 different inequivalent irreducible representations of the group  $\Gamma$ , so we expect to see a quiver diagram with five nodes instead of one as we have written naively above. We will explore the construction of the non-commutative algebraic geometry associated to quiver diagrams in future papers [22, 23], so we will leave this issue at this point.

Our choice of phases has not given the full expected result. Instead, two of the phases select a  $\mathbb{Z}_5^2 \subset \mathbb{Z}_5^3$  subgroup, and the third phase determines the discrete torsion phase for the generators of this subgroup. Thus, the algebra written in (21) is describing a generic  $\mathbb{Z}_5^2$  orbifold of the quintic with discrete torsion.

In a non-generic case we can choose the  $\mathbb{Z}_5^2$  to have singularities in codimension two. In this case we choose  $z_4$  to be in the center of the algebra, and we have only the phase given by  $\alpha$  to worry about. There we see the familiar result for orbifolds with discrete torsion appearing. The branes fractionate in the codimension 2 singularities, and they have the expected monodromies about the codimension three singularities [14].

Now let us consider coordinate transformations to other patches. Indeed, it is not difficult to show that one can still do the standard coordinate changes of the quintic; one must only be careful with ordering of variables. In every coordinate patch the algebra obtained is of the same type as written in (21).

Let us consider the following example,

$$z_i \rightarrow z_1^{-1} z_i \quad (29)$$

As  $z_1^5$  is in the center of the algebra, we may interpret this as

$$z_i \rightarrow \frac{1}{z_1^5} z_1^4 z_i. \quad (30)$$

Thus, denominators may always be taken to lie in the center of the algebra. The ordering of the non-central part in the numerator is irrelevant; the difference between any two choices amounts to a phase, which may be removed by a trivial change of variables.

Algebraically this means that localization takes place in the center. That is, changes of variables can be first done in the center and then lifted to the non-commutative algebra, which fits nicely with the structure proposed in [4] where the center takes a prominent role in the non-commutative algebraic geometry of matrices.

## 5 Outlook

We have shown in this paper that non-commutative algebraic geometric tools are sufficient to describe various compact orbifolds with discrete torsion and their deformations.

The prescription involves gluing various non-commutative holomorphic algebras to compactify spaces. Then, one can calculate the deformations of complex structure by using the

cyclic homology of the glued algebras, and obtain exact agreement with the calculations done by Vafa and Witten [3] with closed string methods.

The non-commutative geometry thus gives a framework in which to interpret backgrounds which do not follow the standard rules of commutative algebraic geometry. In principle, this provides a new set of tools by which one can construct new backgrounds of string theory. These do not have to come from global orbifolds with discrete torsion, and may be singular when viewed from a commutative geometry point of view, while the non-commutative geometry is under control.

The second point which is important is that topological closed string states are associated with cyclic homology. Thus, one should be able to extend this idea beyond the classes of algebras which we have studied. This perspective might shed some light on aspects of string geometry.

The extension of these ideas to more general orbifolds and backgrounds is currently under consideration [22, 23].

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